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COMMENTS ON NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS OF THE LAPLACE EQUATION AND CALCULATION OF EIGENVALUES BY THE GRID METHOD

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COMMENTS ON NUMERICAL SOLUTION OF BOUNDARY VALUE PROBLEMS
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L. A. Lyusternik

Section 1 /49*

Let us assume we must solve the vector equation

$$x = Ax + y, \tag{1}$$

where y is the known; x is the unknown n-dimensional vector; A -- n-dimensional matrix. The method of successive approximations in solving equation (1) consists of the fact that we start with the arbitrary vector x^0 and then subsequently formulate the system of vectors x^1, x^2, ..., x^k,..., where

$$x^k = Ax^{k-1} + y. \tag{2}$$

If the sequence x^k converges to the limiting vector x, then x is the solution of equation (1). However, the sequence convergence may be very slow, and the problem arises of extrapolating the approximations already found of x^1,..., x^k to the root x, so as to approximate it more closely, avoiding the repeated substitution of (2).

Let us examine the simplest case when A is the symmetric matrix. Let us assume λ_1, λ_2,..., λ_n are the eigenvalues of A, and $|\lambda_1| \geqslant |\lambda_2| \geqslant \ldots \geqslant |\lambda_n$ $u_1, u_2, \ldots, u_n$ is the orthonormal system corresponding to it of eigenvectors. Subtracting the equation (1) obtained from (2), we have

*Numbers in margin indicate pagination of original foreign text.

$$x^k - x = A(x^{k-1} - x).$$

(3)

If $x^0 - x = \sum_{i=1}^{n} c_i u_i$, then

$$x^1 - x = A(x^0 - x) = \sum_{i=1}^{n} \lambda_i c_i u_i, \quad x^2 - x = A(x^1 - x) = \sum_{i=1}^{n} \lambda_i^2 c_i u_i;$$

and, in general,

$$x^k - x = \sum_{i=1}^{n} \lambda_i^k c_i u_i.$$

(4)

Repeating the estimates used when determining the eigenvalues /50
numerically using the Mises method, we find: if $|\lambda_1| = |\lambda_2| = \ldots = |\lambda_r| = \lambda, \ |\lambda_{r+j}| < \lambda$ at $j > 0$, then $\left(\frac{\lambda_{r+j}}{\lambda}\right)^k \to 0$ at $k \to \infty$ and

$$x^k - x = \lambda^k \left(\sum_{i=1}^{r} \varepsilon_i c_i u_i + \sum_{i=1}^{n-r} c_{r+j} \left(\frac{\lambda_{r+j}}{\lambda}\right) u_{r+j} \right) \sim \lambda^k \sum_{i=1}^{r} \varepsilon_i c_i u_i, \ |\varepsilon_i| = 1$$

(if one of the numbers $c_1, c_2, \ldots, c_r$ does not equal 0). For large k, the expression $x^k - x$ may be approximately represented by the sum of the eigenvectors corresponding to the eigenvalues which are largest in absolute value. In the case of a symmetric A and, consequently, real eigenvalues, one or two (opposite in sign) eigenvalues (of any multiplicity) may be the largest in absolute magnitude. The criterion for the convergence of the method described of sequential approximations is, as is known, the fact that the eigenvalue of the matrix A which is largest in terms of absolute value is less than unity.

For purposes of simplicity, let us examine the case when there is a singular (of arbitrary multiplicity) eigenvalue of λ which is the largest. Then

$$x^k - x \sim \lambda^k u,$$

(5)

where u is a certain eigenvector corresponding to the eigenvalue of λ. Similarly,

$$x^{k+1} - x \sim \lambda^{k+1} u,$$
$$x^{k+2} - x \sim \lambda^{k+2} u.$$

(6)

Thus

$$
\begin{aligned}
x^k - x^{k+1} &\sim \lambda^k (1-\lambda)\, u, \\
x^k - x^{k+2} &\sim \lambda^k (1-\lambda^2)\, u.
\end{aligned} \tag{7}
$$

The vectors $x^k - x^{k+1}$ and $x^k - x^{k+2}$ for large k are almost proportional, and the ratio of their components approximately equals

$$\frac{1-\lambda^2}{1-\lambda} = 1+\lambda.$$

Thus, if the sequential approximations x^k, x^{k+1}, x^{k+2}, are known, then we may approximately find λ, and since we have the following from (5) and (7)

$$x^k - x \sim \frac{1}{1-\lambda}(x^k - x^{k+1}) \sim \frac{1}{1-\lambda^2}(x^k - x^{k+2}),$$

by determining λ, we may approximately find $x^k - x$, and this also means x.

If λ is very close to unity, then it is advantageous to use similar equations instead of (6)

$$
\begin{aligned}
x^{k+p} - x &\sim \lambda^{k+p} u, \\
x^{k+2p} - x &\sim \lambda^{k+2p} u,
\end{aligned} \tag{6'}
$$

from which we have /51

$$
\begin{aligned}
x^k - x^{k+p} &\sim \lambda^k (1-\lambda^p)\, u, \\
x^k - x^{k+2p} &\sim \lambda^k (1-\lambda^{2p})\, u,
\end{aligned}
$$

and, consequently, the ratio of the components $x^k - x^{k+p}$ and $x^k - x^{k+2p}$ determines $1+\lambda^p$, and this means λ also.

In addition,

$$x^k - x \sim \frac{1}{1-\lambda^p}(x^k - x^{k+p}).$$

Let us now assume operator A has two of the largest eigenvalues for λ and $\lambda_1 = -\lambda$. Then

$$x^k - x \sim \lambda^k [u + (-1)^k u_1] \tag{8}$$

(u and u_1 are the eigenvectors referred to the eigenvalues λ and λ_1), and in addition

$$\begin{aligned} x^{k+2} - x &\sim \lambda^{k+2} [u + (-1)^k u_1], \\ x^{k+4} - x &\sim \lambda^{k+4} [u + (-1)^k u_1], \end{aligned} \tag{9}$$

Therefore, the vectors $x^k - x^{k+2}$ and $x^k - x^{k+4}$ have approximately proportional components, and their ratios approximately equal

$$\frac{1-\lambda^4}{1-\lambda^2} = 1 + \lambda^2,$$

from which we may determine λ and in view of (8) and (9) find x approximately

$$x^k - x \sim \frac{1}{1-\lambda^2}(x^k - x^{k+2}) \sim \frac{1}{1-\lambda^4}(x^k - x^{k+4}).$$

Just as in the previous case, instead of x^k, x^{k+2}, x^{k+4}, we may use x^k, x^{k+2p}, x^{k+4p}.

Section 2

By way of an example, let us give the numerical solution of plane boundary value problem of the Laplace equation by the grid method. Let us first make some preliminary comments.

Let us designate the following operators by ∇, ∇^4 and $D = D^h$

$$\nabla^h z = \frac{1}{h^2}[z(x+h,\ y)+z(x-h,\ y)+$$
$$+z(x,\ y+h)+z(x,\ y-h)-4z(x,\ y)],$$
$$Dz = D^h z = \frac{h^2}{4}\nabla^h z + z = \frac{1}{4}[z(x+h,\ y)+z(x-h,\ y)+$$
$$+z(x,\ y+h)+z(x,\ y-h)],$$
$$\nabla z = \frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}.$$

The operator ∇^n is the finite-difference approximation /52
of the operator ∇. Let us examine a plane square grid with one side of the square having the length h and the finite plane region Q, consisting of the squares of the grid. We shall give the function z for the grid only at the grid corners. Then ∇^h and D^h are changed into finite-dimensional operators (matrices). If the values of z at the corner of the grid (ih, kh), where i,k are whole numbers, can be designated by z_{ik}, then the operator D^h replaces $z_{i,k}$ by the value $\frac{1}{4}(z_{i+1,k}+z_{i-1,k}+z_{i,k+1}+z_{i,k-1})$. Just like the operator ∇, the operator ∇^h has only negative eigenvalues $-\lambda_1^h \geqslant -\lambda_2^h > \dots > -\lambda_n^h > \dots$. The operator D^h has the eigenvalue

$$\mu_i^h = 1 - \frac{h^2}{4}\lambda_i^h, \tag{10}$$

while all $\mu_i^h < 1$.

If the number of internal grid corners at Q equal N and all the functions u, subjected to the operation D^h, have the values 0 at the boundary corners of the grid $\dot{Q}$, then ∇^h and D^h have N difference eigenvalues $-\lambda_1^h, -\lambda_2^h, \dots, -\lambda_N^h;\ \mu_1^h, \mu_2^h, \dots, \mu_N^h$ with the general eigen function $u_1, u_2, \dots, u_n$. The eigen functions u_1, u_2, ..., which corresponds to the first eigenvalues $-\lambda_1^h, -\lambda_2^h, \dots$ approximate the first eigen functions of the Laplace operator ∇ with zero boundary conditions.

We should note that the eigen function $\bar{u}$ of the same operator D^h with an eigenvalue which is inverse in sign corresponds to each of the eigen functions u of the operator D^h, where

$$\bar{u}_{i,k} = (-1)^{i+k} u_{i,k}. \qquad (11)$$

In the same way, if $D^h u = \mu u$, then

$$\frac{1}{4}[u_{i+1,k} + u_{i-1,k} + u_{i,k+1} + u_{i,k-1}] = \mu u_{i,k}, \qquad (12)$$

and since

$$\bar{u}_{i+1,k},\ \bar{u}_{i-1,k},\ \bar{u}_{i,k+1},\ u_{i,k-1}$$

differ from

$$u_{i+1,k},\ u_{i-1,k},\ u_{i,k+1},\ u_{i,k-1}$$

by the factor $(-1)^{k+1} = (-1)^{k-1}$, we have the following from (12)

$$\frac{1}{4}[\bar{u}_{i+1,k} + \bar{u}_{i-1,k} + \bar{u}_{i,k+1} + \bar{u}_{i,k-1}] = -\mu \bar{u}_{ik},$$

i.e., $D^h \bar{u} = -\mu \bar{u}$. The function $\bar{u}$ is the eigen function of the eigenvalue $-\mu$. Therefore, the eigenvalue $-\mu$ of this same operator with the inverse sign corresponds to each eigenvalue μ of the operator D^h. The smallest eigenvalue $\mu_N^h = -\mu_1^h$, corresponds to the largest eigenvalue μ_1^h, and $\mu_{N-1}^h = -\mu_2^h, \ldots;$ corresponds to the eigenvalue μ_2^h. We have

$$1 > \mu_1^h > \mu_2^h > \ldots > \mu_{N-1}^h = -\mu_2^h > \mu_N^h = -\mu_1^h > -1;$$

So that from all $|\mu_i^h| < 1$, we have the well-known fact that the /53
iteration process converges by means of the operator $D = D^h$.

We should note that the eigen functions $u_1, u_2, \ldots$, corresponding to the first eigenvalues μ_1^h, μ_2^h, approximate the first eigen functions of the operator ∇. The eigen functions $\bar{u}_1, \bar{u}_2, \ldots$, determined by means of formula (11), which corresponds to the negative eigenvalues $-\mu_1^h, -\mu_2^h, \ldots$, which are largest in absolute

value, are the singularities of only the grid approximation of the Laplace operator, but not the operator itself. We shall call these eigen functions parasitic. Their existence must be considered when studying the convergence of certain approximation processes.

Section 3

The approximate solution of the boundary value problem for the domain of the Laplace equation by the grid method consists of finding u at the grid corners which satisfies the following equation at the internal corners

$$u = Du \tag{13}$$

and certain boundary value conditions at the boundary corners, for example, finding the values of u at these corners in the case of the Dirichlet problems. For simplicity, let us discuss this problem. Starting with the arbitrary function w at the internal corners of the grid, which uses the given values at the boundary corners, let us formulate the sequence of functions w^1, $w^2, \ldots, w^k, \ldots$, which satisfy the same conditions at the boundary corners, and the internal decisive conditions

$$w^{k+1} = Dw^k.$$

The difference $w^k - w$ satisfies the zero boundary value conditions. If $\bar{u}$ and u are the eigen functions of the operator D, corresponding to the largest and the smallest eigenvalue of μ and $-\mu$, then

$$w^0 - w = cu + \bar{c}\bar{u} + v, \tag{14}$$

where v is the sum of the eigen functions referred to other eigenvalues (if w^0 is selected as a smooth function, then $w^0 - w$ is a smooth function and the coefficient $\bar{c}$, the coefficient with the

parasitic function $\bar{u}$ which changes sign for each pair of adjacent corners, is small). Repeating the previous discussion, we find

$$
\begin{aligned}
&w^k - w \sim \mu^k [cu + (-1)^k \bar{c}\bar{u}],\\
&w^{k+2} - w \sim \mu^{k+2} [cu + (-1)^k \bar{c}\bar{u}],\\
&w^{k+4} - w \sim \mu^{k+4} [cu + (-1)^k \bar{c}\bar{u}],\\
&w^k - w^{k+2} \sim \mu^k (1-\lambda^2) [cu + (-1)^k \bar{c}\bar{u}],\\
&w^k - w^{k+4} \sim \mu^k (1-\lambda^4) [cu + (-1)^k \bar{c}\bar{u}],\\
&w^k - w \sim \frac{1}{1-\lambda^2}(w^k - w^{k+2}) \sim \frac{1}{1-\lambda^4}(w^k - w^{k+4}).
\end{aligned}
\tag{15}
$$

The functions $w^k - w^{k+4}$, $w^k - w^{k+2}$ are approximately proportional, /54
and their ratio approximately equals

$$\frac{1-\lambda^4}{1-\lambda^2} = 1 + \lambda^2.$$

It is best of all to set

$$1 + \lambda^2 = \frac{\sum (w_{i,j}^k - w_{i,j}^{k+4})}{\sum (w_{i,j}^k - w_{i,j}^{k+2})}, \tag{16}$$

where the sums are given for all internal corners of the domain Q. Determining λ, we may use formula (15), from which we may find the approximate value of the function w.

Actually, the calculations are as follows: We find $w^1, \ldots, w^k, \ldots, w^{k+4}$ by the successive use of the operation D. We determine the sums $\sum w_{i,j}^k$, $\sum w_{i,j}^{k+2}$, $\sum w_{i,j}^{k+4}$ for all internal corners, and in addition, we find $1 + \lambda^2$ using formula (16), we determine λ using formula (15) and find w.

Let us examine the example given in the "Handbook for Numerical Solutions of Partial Differential Equations," D. Yu. Panov (GTI, 1943, pp. 50-58) for the solution using the method of successive approximations of the equation Dw = w.

Tables 1, 2 and 3 give the results of the 7, 9 and 11th iterations of the operation D. The values at each corner are located

to the left and above it. The boundary values are only given in Table 1. The values at the corners which are symmetric with respect to the diagonal equal $w^k_{i,j} = w^k_{j,i}$. They are given only on one side of the diagonal ($w^k_{i,j}$ designates the value at the corner of the intersection of the ith low and the jth column after the kth iteration. Tables 1, 2, 3 give the values of $w^7_{i,j}$, $w^9_{i,j}$, $w^{11}_{i,j}$).

Tables 4 and 5 give the differences $w^7_{i,j} - w^{11}_{i,j}$ and $w^7_{i,j} - w^9_{i,j}$ at all corners of the grid. We have

$$\sum (w^7_{i,j} - w^{11}_{i,j}) = 2(0.12 + 0.20 + 0.22 + 0.17 + 0.21 + 0.36 + $$
$$+ 0.39 + 0.27 + 0.43 + 0.24) + 0.1 + 0.33 + 0.48 + 0.41 + 0.14 = 6.68,$$
$$\sum (w^7_{i,j} - w^9_{i,j}) = 2(0.07 + 0.12 + 0.12 + 0.10 + 0.12 + 0.21 + 0.22 + $$
$$+ 0.16 + 0.25 + 0.14) + 0.06 + 0.18 + 0.27 + 0.24 + 0.08 = 3.95.$$

Using the formula (16)

$$1 + \lambda^2 = \frac{6.68}{3.95} = 1.735,$$

from which we have

$$1 - \lambda^2 = 2 - 1.735 = 0.265, \quad \frac{1}{1-\lambda^2} = \frac{1}{0.265} = 3.77.$$

Table 6 gives the values of $w_{i,j} = w^7_{i,j} - 3.77(w^7_{i,j} - w^9_{i,j})$ [see formula (15)]. Table 7 gives the results of applying the averaging operation D to Table 6. Table 8 contains errors in hundreths of unity for the values of Table 8. This accuracy is achieved in the book of D. Panov after 26 iterations. /56

For comparison, in Table 9 we give a table of errors in the values of $w^{26}_{i,j}$ after 26 iterations D according to the "Handbook."

Interpolation used 14 iterations. If we use the formula $w_{i,j} = w^7_{i,j} - \frac{1}{1-\lambda^2}(w^7_{i,j} - w^{11}_{i,j})$, in (15), then the errors are somewhat smaller.

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Table 1 $(w^{0}_{i,j})$

	0	0	0	0	0
15.45	11.99	9.34	7.02	4.73	2.40
29.89	23.00	18.12	13.71	9.34	
40.45	32.30	25.91	19.98		
47.58	31.42	32.82			
52.00	44.76				

Table 2 $(w'_{i,j})$

11.92	9.22	6.90	4.63	2.34
22.88	17.91	13.49	9.16	
32.14	25.66	19.71		
39.23	32.58			
44.68				

Table 3 $(w''_{i,j})$

11.87	9.14	6.80	4.56	2.30
22.79	17.76	13.32	9.01	
32.03	25.48	19.50		
39.18	32.41			
44.62				

Table 4 $(w^{0}_{i,j}-w''_{i,j})$

0.12	0.20	0.22	0.17	0.10
0.21	0.36	0.39	0.33	
0.27	0.43	0.48		
0.24	0.41			
0.14				

Table 5 $(w^{0}_{i,j}-w'_{i,j})$

0.07	0.12	0.12	0.10	0.06
0.12	0.21	0.22	0.18	
0.16	0.25	0.27		
0.14	0.24			
0.08				

Table 6

11.73	8.99	6.57	4.35	2.17
22.58	17.33	12.88	8.66	
31.70	24.97	18.96		
38.89	31.92			
44.46				

Table 7

11.72	8.91	6.53	4.35	2.17
22.54	17.32	12.88	8.62	
31.71	24.98	18.93		
38.91	31.92			
44.45				

Table 8

0	3	3	5	2
1	2	7	6	
−1	−2	5		
−1	−1			
−1				

Table 9

1	2	3	3	1
3	4	5	4	
2	6	7		
3	4			
2				

Section 4

The equation (13) may be replaced by the equivalent equation

$$w = D_\alpha^h w, \qquad (13')$$

where

$$D_\alpha^h w = \frac{1}{1+\alpha}(D^h w + \alpha w).$$

The eigenvalues μ_α^h of the operator D_α^h are connected with the eigenvalues μ^h of the operator D^h by the formula $\mu_\alpha^h = \frac{\mu^h + \alpha}{1+\alpha}$. Since all $|\mu| < 1$, then at $\alpha > 0$ all $|\mu_\alpha| < 1$. The successive use of the operator D_α^h gives a converging process, and to solve the system (13) or (13'), we may use the successive use of the operator D_α^h.

If $\alpha = -\beta < 0$, then

$$\mu_\alpha^h = \frac{\mu^h - \beta}{1 - \beta}.$$

However, since there may be values which are very close to −1 /57
among the eigenvalues μ^h, the eigenvalues μ_α^h may be less than −1. If $-\lambda_1$ is the first eigenvalue of the Laplace operator for our domain, for a small h it is close to the first eigenvalue

$-\lambda_1^h$ for the same domain of the operator $7^h: -\lambda_1^h \sim -\lambda_1$.
In view of (10), we have

$$\mu_1^h \sim 1 - \frac{h^2}{4}\lambda_1; \quad \mu_N^h = \mu_1^h \sim \frac{h^2}{4}\lambda_1 - 1$$

which is very close to -1. The smallest of the $\mu_{\alpha}, \mu_{\alpha N}$ is
$\dfrac{-1-\beta+\frac{h^2}{4}\lambda_1}{1-\beta}$.

For a very small h and a fixed $\alpha = -\beta$, the smallest eigenvalue $\mu_{\alpha N}^h$ becomes less than -1. But then equation (13) cannot be solved by iteration of the operator D_α^h, since it disturbs the necessary convergence condition of the iteration process (but not all of the eigenvalues of D_α^h are less than 1 in absolute value). In this case, the convergence is called the parasitic eigen function. Only the operators D_α^h with positive α can compete with the operator D^h.

The operation $D = D^h$ consists of using the processes of multiplication individually. In terms of simplicity in performing the operation with the massive use of operations, operations which are reduced only to multiplication and division by tens are of interest (i.e., moving the decimal). For example, this is the case with the operation for $\alpha = \frac{1}{5}$, $D_{1/5}^h = \frac{4}{5}(D^h + 1)$

$$D_{1/5}^h z_{ik} = \frac{1}{5}(z_{i,k+1} + z_{i,k-1} + z_{i+1,k} + z_{i-1,k} + z_{i,k}).$$

The operation $D_{1/5}^h$ is reduced to replacing the value of z_{ik} at each grid corner by the mean arithmetic value, and at 5 corners -- by the given value and the 4 adjacent ones. The division of the number by 5 reduces to doubling the number and dividing by 10, i.e., moving the decimal point. The successive use of the operation $D_{1/5}^h$ is actually reduced to the successive use of the operation of multiplication, which may be completely carried out on adding machines (from simple ones to tabulation machines).

If we do not take a square grid, but a rectangular grid, in which the sides are rectangles which are parallel to the z and y axes, referring both to $\sqrt{2}$ and $\sqrt{3}$, then for the approximate solution of the Laplace equation, we must find the function u at the corners of the rectangular grid which satisfies the following equation

$$u = D^h u,$$

where

$$D^h u_{i,k} = \frac{1}{10}\{2u_{i+1,k} + 2u_{i-1,k} + 3u_{i,k+1} + 3u_{i,k-1}\}$$

($u_{i,k}$ is the value of u at the corner, which lies on the intersection of the ith vertical and the kth horizontal. Here the sequential use of the operation is only reduced to summation.

Similarly, in the three-dimensional problem we may introduce the approximate solution of the boundary value Laplace problem by the grid method. One of the methods indicated above, or a combination of them, may reduce all of the successive operations to summation and division by 10, i.e., practically only to summation, and only to addition. For example, let us give this method. /58

Let us consider the following three-dimensional grid: The plane xOy is divided into triangles with equal sides with the side h with three series of parallel lines. The apexes of these triangles are corners of a plane grid. This grid is moved parallel along the z axis and is reproduced on the planes

$$z = z_n = nh\frac{\sqrt{3}}{2}\ (n = \pm 1, \pm 2, \ldots).$$

We obtain a three-dimensional grid, and the function u is given at the grid corners. Let us assume A is a grid corner (apex of the triangle) in the plane $z = z_n$.

Let us introduce the operation F^h, which changes the value of u(A) at the corner A, divided by 10, and the sum of the value u at 6 corners of the same plane, which are adjacent to A, and the double

sum of the value u at 2 corners lying above and below the point A in the plane $z = z_{n+1}$ and $z = z_{n-1}$.

As may be readily seen, the approximate solution of the Laplace equation reduces to find the solution of the equation $u = F^h u$ at the grid corners. The solution of this equation by the method of successive approximation is reduced to the successive use of the operation F^h, i.e., only to summation.

Let us assume we must find the three-dimensional solution of the Laplace equation with axial asymmetry. The axis of symmetry is with the z axis, and we assume that the previous grid is constructed so that one of the triangle apexes in the $z = 0$ plane (and in the parallel plane $z = z_n$) passes through the symmetry axis.

The function u must assume equal values at the corners of each plane $z = z_u$, which are equally removed from the axis. Since our grid has 6 symmetry planes passing through the axis, the number of corners corresponding to the different values of u is approximately 6 times less than the number of corners of the three-dimensional domain, and it approximates a plane grid using the number of unknown values of the three-dimensional problem thus solved.

Section 5

Let us examine the Sturm-Liouville equation

$$\frac{d}{dx}[R(x)y'(x)] - P(x)y(x) = \lambda y(x) \qquad (17)$$

under the boundary value conditions, for example $y(a)=0$, $y(b)=0$. We will have $R(x) > 0$ at the segment $[a, b]$. We shall also assume that on $[a, b]$ $P(x) > 0$.

Along with (1), let us consider the equation

$$\frac{d}{dx}\left[R(x)\frac{\partial w(x,t)}{\partial x}\right] - P(x)\,w(x,t) = \frac{\partial w(x,t)}{\partial t}. \tag{18}$$

Let us assume $u^1(x), u^2(x), \ldots, u^k(x), \ldots$ is a sequence of eigen /59
functions of the equation (17), corresponding to the values $\lambda_1 > \lambda_2 > \lambda_3 > \ldots \lambda_k > \ldots$; if $P(x) \leqslant 0$ on $[a, b]$, then λ_i are negative. If

$$\varphi(x) = \sum_{k=1}^{\infty} c_k u^k(x), \tag{19}$$

then the solution of (18) with the initial value of $w(x,0) = \varphi(x)$ is

$$w(x,t) = \sum_{k=1}^{\infty} c_k u^k(x) e^{\lambda_k t}. \tag{20}$$

Since all $\lambda_k < 0$, then $w(x,t)$ approximately strives to zero at $t \to \infty$, and the terms $c_k u^k(x) e^{\lambda_k t}$ strive to zero more rapidly, the larger is k. If, for example, $c_1 \neq 0$, then for a very large t the first term $w(x,t) \sim c_1 u^1(x) e^{\lambda_1 t}$. will be the main term in (19).

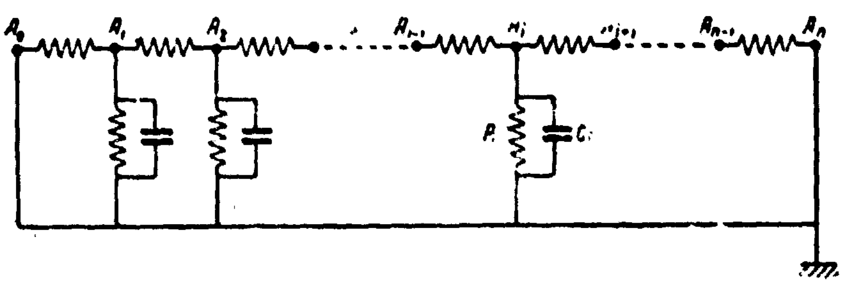

Fig. 1

For a fixed rather large t, the function $w(x,t)$ may become approximately proportional to the first eigen function $u_1^1(x)$, with the relative accuracy which increases as t increases. If we simulate the equation (18), then we may find the approximate values of the eigen functions of the equation (17), and the eigenvalues.

The equation (18) may be modeled for a linear electric circuit of resistances and capacitances.

Dividing the segments [a,b] into n equal parts with the length $h=\frac{b-a}{n}$ and setting $f(a+ih)=f_i$, we replace equation (17) by its finite difference analog

$$\frac{1}{h^2}[R_i(y_{i+1}-y_i)-R_{i-1}(y_i-y_{i-1})]-P_i y_i=\lambda y_i. \tag{21}$$

Along with (21), let us consider a system of ordinary equations which approximate the partial differention equation (18)

$$\frac{1}{h^2}[R_i(w_{i+1}(t)-w_i(t))+R_{i-1}(u_i(t)-u_{i-1}(t))]-P_i u_i(t)=Cw_i'(t). \tag{22}$$

All of the previous considerations hold if we designate the "eigen functions" of equation (21) by u^k ($k = 1, 2, \ldots$). These functions are given only at the points a = ih, and we use λ_k to /60
designate the corresponding eigenvalues which approximate the eigen functions and values of the initial equation (1) at $h \to 0$.

Let us formulate a linear scheme of resistances and capacitances with the corners A_i ($i = 0, 1, 2, \ldots, n$) (Fig. 1). Each corner A_i is connected with the corners A_{i-1} and A_{i+1} by the conductivities R_{i-1} and R_i, and currents with the conductivity P_i and the capacitance C emanate from it. The voltage equals 0 at the corners A_o and A_n. The voltage at the corners A_i ($i = 1, 2, \ldots, n-1$) satisfies equation (22). At the corners $A_1, A_2, \ldots, A_i, \ldots, A_{n-1}$ setting the arbitrary system of voltage, in view of the previous statements, after a certain time interval t, we obtain a system of voltages $w_i(t)$ which is approximately proportional to the system of values for the first eigen function u^1 at the points a + ih. If we give the voltage in hundreths of volts and measure them at the time t_o for a voltage of about one volt, then the measured voltages at the corners A_i give the approximate values of the first eigen function at the points a + ih. Thus, the sign and overtones -- of other eigen functions, are retained. If the measured values of the voltage -- when we multiply them, let us say, by 100 -- are given at the corners A_i and are again

measured in the time interval t_o, then we may obtain the more "pure" value of the first eigen function (with weaker traces of the higher eigen functions).

Let us assume we have obtained the first eigen function u_i^1 which assumes the value u_i^1 at the points $A_i (i = 1, 2, \ldots, n - 1)$. Let us now give the system of voltages $\bar{w}$ at the corners A_i, so that

$$\sum \bar{w}_i \cdot u_i^1 = 0,$$

i.e., the initial distribution represents the function $\bar{w}$ which is orthogonal to the first eigen function u'. For example, let us assume that at all of the corners except two, $\bar{w} = 0$, and at the corners A_j and A_k

$$\bar{w}_j : \bar{w}_k = - u_k^1 : u_j^1.$$

Similar measurements would have to give the second eigen function u^2. The given function w is orthogonal to u', and therefore the coefficient $c_1 = 0$ in the representation of (3) when ϕ is replaced by $\bar{w}$. Now, with an increase in t, the second term will decrease the most slowly

$$c_2 u^2 (x) e^{\lambda_2 t}$$

with the following eigen function u^2.

The coefficient c_1 does not precisely equal zero for u', although it is small, but since the first term decreases more slowly than the others, the measured system of values at the time t_o does not give an expression which is approximately proportional to the second eigen function. Generally speaking, it may be regarded as a linear combination of the first two eigen functions. However, the first eigen system is known. Therefore, we may determine the /61
second from this linear combination.

The transition to the third eigen function is similar.

Let us assume $P(x) \geqslant 0$ in the interval (a, b). If this condition is violated, then it is sufficient to replace the coefficient P(x) in (17) by the coefficient $\bar{P}(x) = P(x) + C$ ($C = \text{const}$) so that it is positive in this interval. This substitution leaves the eigen functions of equation (17) unchanged, and decreases all of its eigenvalues by the number C.

We may use a similar method to model the eigenvalues for several plane and three-dimensional self-conjugate equations -- the linear system is changed to be plane or three-dimensional (for example, finding the eigen functions of the equation $\nabla u = \lambda u$ etc.

Note. Let us assume we are solving an experimental problem of finding the eigenvalues and the eigenvectors for the equation

$$Ay = \lambda y$$

(y is the n-dimensional vector; A - square matrix). The solution of this problem may be modeled very simply if the diagonal terms have one sign in the matrix A. This can be achieved by adding the corresponding positive constant C to the diagonal terms. In addition, the eigenvectors are not changed, and the eigenvalues increase by C [our equation is replaced by the equation $(A + C)y = (\lambda + C)y$].

We give this in the problem examined.

The answers obtained will contain inaccuracies following both from the unreliable estimate of the measurements, and from the distortion which we assume, as well as by the replacing of the differential operator by the finite-difference operator. For a plane problem to be solved by the grid method, we may replace an arbitrary contour by a polygon contour. However, we may greatly improve the results by regarding the distortions we introduced as perturbations of the operator and by determining the influence of these

perturbations on the result obtained. This pertains not only to the method given, but to any other experimental or numerical method of solving these problems.

Section 6

Let L be a self-conjugate operator and we shall approximately solve the operator equation

$$Lu - \lambda u = 0 \tag{23}$$

under the normalization condition

$$(u, u) = 1. \tag{24}$$

Actually, we replace the operator L by the approximating operator L_1 (we replace the differences by the derivative in the differential operator, the integrals by the sums in the integral operator, etc.), where $L_1 = L + \delta L$ (in the previous example δ L is the difference between the differential and the finite difference operators, which is expressed by means of the difference of higher orders).

We have found the approximate eigenvalue λ_1 and the approximate normalized eigen element u_1 /62

$$L_1 = L + \delta L, \quad u_1 = u + \delta u,$$

which satisfy the equation

$$L_1 u_1 - \lambda_1 u = \varepsilon \tag{25}$$

(the nonzero right side is determined by calculation errors, measurement errors, etc.).

Substituting (23) from (25) and disregarding terms of the second order of magnitude with respect to $\delta \lambda$ and δ u, i.e., replacing the difference of the first variation, we obtain

$$L\delta u + \delta L u - \lambda\delta u - \delta\lambda u = \varepsilon. \qquad (26)$$

Multiplying both sides of (26) by u and using the equations (23) and (24), and in addition, due to the self-conjugate nature of L, we obtain

$$(Lu - \lambda u)\delta u + (\delta L u, u) - \delta\lambda = (\varepsilon, u),$$

from which we have

$$\delta\lambda = (\delta L u, u) - (\varepsilon, u). \qquad (27)$$

The first term on the right side is the regular expression for the perturbation of the eigenvalue. The second is determined by the inaccuracies of solving the "perturbed" equation. Finding $\delta\lambda$ and designating λ_1 by $\gamma_1 - \delta\lambda$, we obtain a more precise value of λ. Equation (26), which is solved with a relatively small accuracy, determines the correction δ_u.

In problems for finding the eigenvalues of the differential operators, the boundary values are included -- the zero values at the boundary. Let us use the symbol δ_2 to designate the variation caused by varying the boundary. For an approximate solution, if we assume the distortion of the boundary (for example, in the plane problem we replace the curvilinear plane contour by a contour with several corners), then the term $\delta_2\lambda$ appears in the right side of (27) -- the variation of the eigenvalue caused by deformation of the boundary.

R. Courant developed (see, for example, Part 1 of "Methods of Mathematical Physics") a variational theory of eigenvalues based on the method of the minimum-maximum. However, this theory gives only a qualitative estimate -- the increase or decrease of the eigenvalues, i.e., only the sign of $\delta_2\lambda$, while the regular elementary methods of the variational value give the values $\delta_2\lambda$,

i.e., they determine (within an accuracy of the higher variations) the numerical value of the distortion introduced by the small deformation of the boundary, and make it possible to refine the result.

Let us assume L is the self-conjugate differential operator with certain zero boundary data.

Let us use $J(u)=(Lu,u)$, to designate $J_1(u)=J(u)-\lambda(u,u)$. Equation (23) is the Euler equation $\delta J(u)=0$. Under the isoperimetric condition (u,u)= 1 or $\delta J_1(u)=0$. The normalized eigen /63
function is the extremum of the equation $\delta J_1(u)=0$, and the eigenvalue $\lambda=J(u,u)$.

We shall vary the boundary. Then $\delta_2\lambda=\delta_2 J$. Since the normalization condition retains its form,

$$\delta_2(u,u)=0, \quad \text{т. е.} \quad \delta_2\lambda=\delta_2 J=\delta_2 J_1.$$

The last side of this equation is calculated on the basis of the regular rules for the variation of the functional from the extremum with a variation of the boundary.

<u>Example 1</u>. L is the Sturm-Liouville operator. We have the equation (17) under the condition u(a) = u(b) = 0 and the normalization

$$(u,u)=\int_a^b u^2\,dx=1.$$

Here

$$J(u)=(Lu,u)=\int_a^b (Ru'^2+Pu^2)\,dx,$$

$$J_1(u)=\int_a^b (Ru'^2+Pu^2-\lambda u^2)\,dx.$$

Setting $F=Ru'^2+Pu^2-\lambda u^2$, we have

$$F_{u'} = 2Ru', \quad F - u'F_{u'} = -Ru'^2 + Pu^2 - \lambda u^2.$$

If the finite abscissas a and b obtain the variation λa and λb,

$$\delta_2\lambda = \delta_2 J_1 = -(F - u'F_{u'})_{x=a}\,\delta a + (F - u'F_{u'})_{x=b}\,\delta b.$$

At the point square x = a, x = b, we have

$$F - u'F_{u'} = -Ru'^2,$$

Therefore

$$\delta_2\lambda = R(a)[u'(a)]^2\,\delta a - R(b)[u'(b)]^2\,\delta b. \qquad (28)$$

Since

$$R(a) > 0, \quad R(b) > 0,$$

then at $\delta a > 0, \delta b < 0$ [i.e., with compression of the integral (a,b)]

$$\delta\lambda_2 > 0,$$

i.e., the eigenvalues increase. Thus, the qualitative side of the variational series of eigenvalues is completely obtained. Formula (28) gives the quantitative estimate.

Example 2. Let us assume L = ∇ is the plane Laplace operator applied to the functions u(x,y) given in the plane domain Q /64 with zero boundary values at the boundary q of the domain Q. Then

$$J(u) = \iint_Q (\nabla u)\, u\, dx\, dy = \iint_Q \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] dx\, dy,$$

$$J_1(u) = \iint_Q \left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 - \lambda u^2\right] dx\, dy.$$

Let us vary the boundary q in the vicinity of its point A, and we assume $\delta\sigma_A$ is the variation of the area Q. Then

$$\delta_2 J_1(u) = -[\text{grad}\, u(A)]^2\, \delta\sigma_A.$$

and, consequently,

$$\delta_2 \lambda = -[\operatorname{grad} u(A)]^2 \delta\sigma_A. \tag{29}$$

Thus, the qualitative estimate of $\delta_2 \lambda$ is obtained: at $\delta\sigma_A < 0$, $\delta_2\lambda > 0$, i.e., the eigenvalues of the operator ∇ increase, if the domain Q is reduced, changing into the eigen part. However, formula (28) gives the quantitative estimate.

Let us assume we have found the eigenvalue for λ and the eigen functions of u for the domain Q_1 which approximates the domain Q (for example, for the domain consisting of the quadrile squares, if we solve experimentally the problem for the plane square grid). Let us divide the band between Q and Q_1 into small parts of the area $\delta\sigma_{A_i}$, $i=1, 2, \ldots, n$, which are adjacent to the point A_i, i = 1, 2, ..., n of the boundary q. With the transition from Q_1 to Q, the eigenvalue λ receives the increment $\delta_2\lambda$ which approximately equals

$$\delta_2 \lambda \approx -\sum_i [\operatorname{grad} u(A_i)]^2 \delta\sigma_{A_i}.$$